

STRONG SIGN-COHERENCY OF CERTAIN SYMMETRIC POLYNOMIALS, WITH APPLICATION TO CLUSTER ALGEBRAS

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ABSTRACT. For each positive integer n , we define a polynomial in the variables z_1, \dots, z_n with coefficients in the ring $\mathbb{Q}[q, t, r]$ of polynomial functions of three parameters q, t, r . These polynomials naturally arise in the context of cluster algebras. We conjecture that they are symmetric polynomials in z_1, \dots, z_n , and that their expansions in terms of monomial, Schur, complete homogeneous, elementary and power sum symmetric polynomials are sign-coherent.

1. INTRODUCTION

The purpose of this note is to introduce an interesting family of (conjecturally symmetric) polynomials, which naturally arises in the context of cluster algebras. For each positive integer n , we define a polynomial in the variables $Z = \{z_1, \dots, z_n\}$ with coefficients in the ring $\mathbb{Z}[q, t, r]$ of polynomial functions of the three parameters q, t, r . In Section 2, we will explain their connection to cluster algebras. In order to define them, we need some notations.

Let $S = \{1, 2, 3, \dots, n\}$, and \mathcal{S} be the collection of all set partitions of S . For any subset S_j , say $\{v_1, \dots, v_m\}$, of S , let

$$\sigma(S_j) := z_{v_1} + \dots + z_{v_m}.$$

When $P = S_1 \sqcup S_2 \sqcup \dots \sqcup S_k$ is a partition of S , we denote k by $|P|$. We give an order on $\{S_1, \dots, S_k\}$ as follows : $S_i < S_j$ if and only if $\min S_i < \min S_j$. For any partition of S , we assume that $S_1 < \dots < S_k$. Let

$$d(j, i) := |S_j| \sigma(S_i) - |S_i| \sigma(S_j).$$

Definition 1. Let $T : \{1, 2, \dots, m\} \rightarrow \mathbb{N}$ be any strictly increasing function. Let \mathcal{W} be the set of all permutations of $\{2, \dots, m\}$. We define $e(T)$ by

$$e(T) = \begin{cases} 1, & \text{if } m = 1 \\ \frac{1}{m!} \sum_{(p_2, \dots, p_m) \in \mathcal{W}} \prod_{i=1}^{m-1} (r(z_{T(1)} + z_{T(p_2)} + \dots + z_{T(p_i)} - iz_{T(p_{i+1})}) - i), & \text{otherwise.} \end{cases}$$

Let S_j be any subset, say $\{v_1 < \dots < v_m\}$, of $S = \{1, 2, 3, \dots, n\}$. It naturally gives rise to the increasing function T , that is $T(i) = v_i$ ($1 \leq i \leq m$). By abuse of notation, set $e(S_j) = e(T)$. For any partition $P (= S_1 \sqcup S_2 \sqcup \dots \sqcup S_k)$ of S , we define

$$e(P) := \prod_{j=1}^k e(S_j).$$

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□

Example 2.

$$\begin{aligned}
e(\{1, 2, 4\} \sqcup \{3, 5\}) &= e(\{1, 2, 4\})e(\{3, 5\}) \\
&= \frac{1}{3!}((r(z_1 - z_2) - 1)(r(z_1 + z_2 - 2z_4) - 2) + (r(z_1 - z_4) - 1)(r(z_1 + z_4 - 2z_2) - 2)) \\
&\quad \times \frac{1}{2!}(r(z_3 - z_5) - 1).
\end{aligned}$$

Now we are ready to define our polynomial.

Definition 3.

$$EC_n(Z; q, t, r) := n! \sum_{P(=S_1 \sqcup S_2 \sqcup \dots \sqcup S_{|P|}) \in \mathcal{S}} e(P) \prod_{j=1}^{|P|} \left(-|S_j|q + t\sigma(S_j) - r \sum_{i=1}^{j-1} d(j, i) \right).$$

We call this the ‘*EC*’-polynomial, because it naturally occurs in the expression of the ‘Euler Characteristic’ of a cell of a certain quiver Grassmannian. Despite its non-symmetric appearance, the *EC*-polynomial is expected to be symmetric in z_1, \dots, z_n . Furthermore, it seems to have surprising positivity properties.

Conjecture 4. *For every positive integer n , $EC_n(Z; q, t, r)$ is a symmetric polynomial of z_1, \dots, z_n .*

Conjecture 5. *Suppose that Conjecture 4 is true. When expanded in terms of monomial (resp. Schur, complete homogeneous, elementary and power sum) symmetric polynomials, $EC_n(Z; q, t, r)$ is sign-coherent, i.e., for any partition λ , its coefficient of m_λ (resp. $s_\lambda, h_\lambda, e_\lambda, p_\lambda$) is a polynomial of q, t, r with either all nonnegative or all nonpositive integer coefficients, and the sign depends only on the parity of the number of parts in λ .*

Conjecture 4 and Conjecture 5 are true for $n \leq 7$.

Example 6.

$$EC_1(Z; q, t, r) = e(\{1\})(-q + tz_1) = -q + tz_1.$$

$$\begin{aligned}
&EC_2(Z; q, t, r) \\
&= 2e(\{1\} \sqcup \{2\})(-q + tz_1)(-q + tz_2 - r(z_1 - z_2)) + 2e(\{1, 2\})(-2q + t(z_1 + z_2)) \\
&= 2e(\{1\})e(\{2\})(-q + tz_1)(-q + tz_2 - r(z_1 - z_2)) + 2e(\{1, 2\})(-2q + t(z_1 + z_2)) \\
&= 2(-q + tz_1)(-q + tz_2 - r(z_1 - z_2)) + 2 \frac{r(z_1 - z_2) - 1}{2}(-2q + t(z_1 + z_2)) \\
&= -tr(z_1^2 + z_2^2) + (2t^2 + 2tr)z_1z_2 + (-2qt - t)(z_1 + z_2) + 2q^2 + 2q.
\end{aligned}$$

$$\begin{aligned}
& EC_3(Z; q, t, r) \\
&= 6e(\{1\} \sqcup \{2\} \sqcup \{3\})(-q + tz_1)(-q + tz_2 - r(z_1 - z_2))(-q + tz_3 - r(z_1 - z_3 + z_2 - z_3)) \\
&\quad + 6e(\{1, 2\} \sqcup \{3\})(-2q + t(z_1 + z_2))(-q + tz_3 - r(z_1 + z_2 - 2z_3)) \\
&\quad + 6e(\{1, 3\} \sqcup \{2\})(-2q + t(z_1 + z_3))(-q + tz_2 - r(z_1 + z_3 - 2z_2)) \\
&\quad + 6e(\{1\} \sqcup \{2, 3\})(-q + tz_1)(-2q + t(z_2 + z_3) - r(2z_1 - z_2 - z_3)) \\
&\quad + 6e(\{1, 2, 3\})(-3q + t(z_1 + z_2 + z_3)) \\
&= 2tr^2(z_1^3 + z_2^3 + z_3^3) + (-3t^2r - 3tr^2)(z_1^2z_2 + \cdots + z_3^2z_1) + (6t^3 + 18t^2r + 12tr^2)z_1z_2z_3 \\
&\quad + (6qtr + 6tr)(z_1^2 + z_2^2 + z_3^2) + (-6qt^2 - 6qtr - 6t^2 - 6tr)(z_1z_2 + z_2z_3 + z_3z_1) \\
&\quad + (6q^2t + 12qt + 4t)(z_1 + z_2 + z_3) - 6q^3 - 18q^2 - 12q \\
&= 2tr^2s_{(3)}Z + (-3t^2r - 5tr^2)s_{(2,1)}Z + (6t^3 + 24t^2r + 20tr^2)s_{(1,1,1)}Z \\
&\quad + (6qtr + 6tr)s_{(2)}Z + (-6qt^2 - 12qtr - 6t^2 - 12tr)s_{(1,1)}Z \\
&\quad + (6q^2t + 12qt + 4t)s_{(1)}Z - 6q^3 - 18q^2 - 12,
\end{aligned}$$

where $s_\lambda Z$ are Schur symmetric polynomials. □

2. THE EULER CHARACTERISTIC OF QUIVER GRASSMANNIANS

In this section, we explain how EC -polynomials arise in the context of cluster algebras.

First, we define cluster algebras. To avoid too much distraction, we restrict ourselves to the rank 2 case. Let b, c be positive integers and x_1, x_2 be indeterminates. The (coefficient-free) *cluster algebra* $\mathcal{A}(b, c)$ is the subring of the field $\mathbb{Q}(x_1, x_2)$ generated by the elements x_m , $m \in \mathbb{Z}$ satisfying the recurrence relations:

$$x_{n+1} = \begin{cases} (x_n^b + 1)/x_{n-1} & \text{if } n \text{ is odd,} \\ (x_n^c + 1)/x_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

The elements x_m , $m \in \mathbb{Z}$ are called the cluster variables of $\mathcal{A}(b, c)$. Fomin and Zelevinsky [2] introduced cluster algebras and proved the Laurent phenomenon whose special case says that for every $m \in \mathbb{Z}$ the cluster variable x_m can be expressed as a Laurent polynomial of $x_1^{\pm 1}$ and $x_2^{\pm 1}$. In addition, they conjectured that the coefficients of monomials in the Laurent expression of x_m are non-negative integers. When $bc \leq 4$, Sherman-Zelevinsky [6] and independently Musiker-Propp [5] proved the conjecture. Moreover in this case the explicit combinatorial formulas for the coefficients are known. In [4], we find a new formula for the coefficients when $b = c \geq 2$.

Before we state the main results of [4], we need some definitions.

Definition 7. For arbitrary (possibly negative) integers A, B , we define the modified binomial coefficient as follows.

$$\left[\begin{matrix} A \\ B \end{matrix} \right] := \begin{cases} \prod_{i=0}^{A-B-1} \frac{A-i}{A-B-i}, & \text{if } A > B \\ 1, & \text{if } A = B \\ 0, & \text{if } A < B. \end{cases}$$

□

If $A \geq 0$ then $\left[\begin{matrix} A \\ B \end{matrix} \right] = \left[\begin{matrix} A \\ A-B \end{matrix} \right]$ is just the usual binomial coefficient. In general, $\left[\begin{matrix} A \\ A-B \end{matrix} \right]$ is equal to the generalized binomial coefficient $\binom{A}{B}$. But in this paper we use our modified binomial coefficients to avoid too complicated expressions.

Definition 8. Let $\{a_n\}$ be the sequence defined by the recurrence relation

$$a_n = ca_{n-1} - a_{n-2},$$

with the initial condition $a_1 = 0, a_2 = 1$. If $c = 2$ then $a_n = n - 1$. When $c > 2$, it is easy to see that

$$a_n = \frac{1}{\sqrt{c^2-4}} \left(\frac{c + \sqrt{c^2-4}}{2} \right)^{n-1} - \frac{1}{\sqrt{c^2-4}} \left(\frac{c - \sqrt{c^2-4}}{2} \right)^{n-1} = \sum_{i \geq 0} (-1)^i \binom{n-2-i}{i} c^{n-2-2i}.$$

□

Our main result in [4] is the following.

Theorem 9. Assume that $b = c \geq 2$. Let $n \geq 3$. Then

$$(2.1) \quad x_n = x_1^{-a_{n-1}} x_2^{-a_{n-2}} \sum_{e_1, e_2} \sum_{t_0, t_1, \dots, t_{n-4}} \left[\left(\prod_{i=0}^{n-4} \left[\begin{matrix} a_{i+1} - cs_i \\ t_i \end{matrix} \right] \right) \right. \\ \left. \times \left[\begin{matrix} a_{n-2} - cs_{n-3} \\ a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \end{matrix} \right] \left[\begin{matrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - e_1 + s_{n-3} \end{matrix} \right] x_1^{c(a_{n-2}-e_2)} x_2^{ce_1} \right],$$

where

$$s_i = \sum_{j=0}^{i-1} a_{i-j+1} t_j,$$

and the summations run over all integers $e_1, e_2, t_0, \dots, t_{n-4}$ satisfying

$$(2.2) \quad \begin{cases} 0 \leq t_i \leq a_{i+1} - cs_i \ (0 \leq i \leq n-4), \\ 0 \leq a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \leq a_{n-2} - cs_{n-3}, \text{ and} \\ e_2 a_{n-1} - e_1 a_{n-2} \geq 0. \end{cases}$$

Since $\left[\begin{matrix} A \\ B \end{matrix} \right] \neq 0$ if and only if $A \geq B$, we may add the condition $0 \geq -e_1 + s_{n-3}$ to (2.2). Then the summation in the statement is guaranteed to be a finite sum. A referee

remarks that F -polynomials have similar expressions. As he pointed out, the expression without (2.2) is an easy consequence of the formula (6.28) in the paper [3] by Fomin and Zelevinsky, and the one with $e_2 a_{n-1} - e_1 a_{n-2} \geq 0$ is a consequence of [6, Proposition 3.5] in the paper by Sherman and Zelevinsky. Our contribution is to show that all the modified binomial coefficients in (2.1) except for the last one are non-negative.

As a corollary to Theorem 9, we obtain a new expression for the Euler-Poincaré characteristic of the variety $\text{Gr}_{(e_1, e_2)}(M(n))$ of all subrepresentations of dimension (e_1, e_2) in a unique (up to an isomorphism) indecomposable Q_c -representation $M(n)$ of dimension (a_{n-1}, a_{n-2}) , where Q_c is the generalized Kronecker quiver with two vertices 1 and 2, and c arrows from 1 to 2. We use a result of Caldero and Zelevinsky [1, Theorem 3.2 and (3.5)].

Theorem 10 (Caldero and Zelevinsky). *The cluster variable x_n is equal to*

$$x_1^{-a_{n-1}} x_2^{-a_{n-2}} \sum_{e_1, e_2} \chi(\text{Gr}_{(e_1, e_2)}(M(n))) x_1^{c(a_{n-2}-e_2)} x_2^{ce_1}.$$

Corollary 11. *Assume that $b = c \geq 2$. For any (e_1, e_2) and $n \geq 3$, the Euler-Poincaré characteristic of $\text{Gr}_{(e_1, e_2)}(M(n))$ is equal to*

$$(2.3) \quad \sum_{t_0, t_1, \dots, t_{n-4}} \left[\left(\prod_{i=0}^{n-4} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \begin{bmatrix} a_{n-2} - cs_{n-3} \\ a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \end{bmatrix} \begin{bmatrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - e_1 + s_{n-3} \end{bmatrix} \right],$$

where the summation runs over all integers t_0, \dots, t_{n-4} satisfying

$$(2.4) \quad \begin{cases} 0 \leq t_i \leq a_{i+1} - cs_i \ (0 \leq i \leq n-4), \text{ and} \\ 0 \leq a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \leq a_{n-2} - cs_{n-3}. \end{cases}$$

Proof. Corollary 11 is an immediate consequence of Theorem 9 thanks to the result of Caldero and Zelevinsky [1, Theorem 3.2 and (3.5)]. \square

Corollary 12. *Assume that $b = c \geq 3$. Let $n \geq 3$. For any (e_1, e_2) with $e_2 \geq \frac{a_{n-3}}{c}$, the Euler-Poincaré characteristic of $\text{Gr}_{(e_1, e_2)}(M(n))$ is non-negative.*

Proof. By (2.4), all the modified binomial coefficients except for the last one in (2.3) are non-negative. If $e_2 \geq \frac{a_{n-3}}{c}$ then the last one also becomes non-negative. Therefore, Corollary 11 implies that $\chi(\text{Gr}_{(e_1, e_2)}(M(n)))$ is non-negative. \square

In order to prove (or disprove) that the Euler characteristic of $\text{Gr}_{(e_1, e_2)}(M(n))$ is non-negative for $0 < e_2 < \frac{a_{n-3}}{c}$, we need to find another expression for the Euler characteristic, preferably an expression which could explain a cell decomposition of $\text{Gr}_{(e_1, e_2)}(M(n))$. Conjecturally we have a better expression for this purpose, especially when e_1 is small.

Lemma 13. *Assume that $b = c \geq 2$. If $e_1 < c$ and $n \geq 4$, then the Euler-Poincaré characteristic of $\text{Gr}_{(e_1, e_2)}(M(n))$ is equal to*

$$(2.5) \quad \sum_{t_{n-4}} \begin{bmatrix} a_{n-3} \\ t_{n-4} \end{bmatrix} \begin{bmatrix} a_{n-2} - ct_{n-4} \\ a_{n-2} - ct_{n-4} - e_2 \end{bmatrix} \begin{bmatrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - e_1 + t_{n-4} \end{bmatrix}.$$

Proof. By Corollary 11, the Euler characteristic is equal to (2.3), where $t_i \geq 0$ ($0 \leq i \leq n-4$). If $t_i \geq 1$ for some $0 \leq i \leq n-5$, then $s_{n-3} = \sum_{j=0}^{n-4} a_{n-2-j} t_j \geq a_3 = c > e_1$, which implies $\begin{bmatrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - e_1 + s_{n-3} \end{bmatrix} = 0$. So we can assume that $t_i = 0$ for $0 \leq i \leq n-5$. Then we have $s_{n-4} = 0$, $s_{n-3} = t_{n-4}$, and all the modified binomial coefficients except for the last three are 1. Therefore, (2.3) reduces to (2.5). \square

We will need the following standard fact later, whose proof will be omitted.

Lemma 14. *Let A, B be any (possibly negative) integers, and let n be any positive integer. Then*

$$\frac{d^n}{dy^n} (1 + y^A)^B = n! \sum_{i=1}^n \sum_{\substack{j_1 + \dots + j_i = n \\ j_1, \dots, j_i \geq 1}} \begin{bmatrix} B \\ B-i \end{bmatrix} (1 + y^A)^{B-i} \begin{bmatrix} A \\ A-j_1 \end{bmatrix} \cdots \begin{bmatrix} A \\ A-j_i \end{bmatrix} y^{Ai-n}.$$

Lemma 15. *Assume that $b = c \geq 2$. If $e_1 < c$ and $n \geq 4$, then the Euler-Poincaré characteristic of $\text{Gr}_{(e_1, e_2)}(M(n))$ is equal to*

$$(2.6) \quad \binom{ce_2}{e_1} \binom{a_{n-2}}{e_2} + \sum_{k=1}^{e_2} \sum_{i=1}^k \sum_{\substack{j_1 + \dots + j_i = k \\ j_1, \dots, j_i \geq 1}} \binom{a_{n-3}}{i} \binom{ce_2 - i}{e_1 - i} \begin{bmatrix} -c \\ -c - j_1 \end{bmatrix} \cdots \begin{bmatrix} -c \\ -c - j_i \end{bmatrix} \binom{a_{n-2}}{e_2 - k}.$$

Proof. We want to show that (2.5) is equal to (2.6). We start with the following binomial formula:

$$(1 + y^{-c})^{a_{n-3}} y^{a_{n-2}} = \sum_i \binom{a_{n-3}}{i} y^{a_{n-2} - ci}.$$

By taking the e_2 -th derivative, we get

$$(2.7) \quad \frac{1}{e_2!} \frac{d^{e_2}}{dy^{e_2}} [(1 + y^{-c})^{a_{n-3}} y^{a_{n-2}}] = \sum_i \binom{a_{n-3}}{i} \begin{bmatrix} a_{n-2} - ci \\ a_{n-2} - ci - e_2 \end{bmatrix} y^{a_{n-2} - ci - e_2}.$$

Then we multiply (2.7) by

$$(1 + y^{-c})^{-a_{n-3} + e_2 c} = \sum_j \begin{bmatrix} -a_{n-3} + e_2 c \\ -a_{n-3} + e_2 c - e_1 + j \end{bmatrix} (y^{-c})^{e_1 - j},$$

which yields

$$(2.8) \quad \begin{aligned} & (1 + y^{-c})^{-a_{n-3} + e_2 c} \frac{1}{e_2!} \frac{d^{e_2}}{dy^{e_2}} [(1 + y^{-c})^{a_{n-3}} y^{a_{n-2}}] \\ &= \sum_i \binom{a_{n-3}}{i} \begin{bmatrix} a_{n-2} - ci \\ a_{n-2} - ci - e_2 \end{bmatrix} y^{a_{n-2} - ci - e_2} \sum_j \begin{bmatrix} -a_{n-3} + e_2 c \\ -a_{n-3} + e_2 c - e_1 + j \end{bmatrix} (y^{-c})^{e_1 - j} \\ &= \sum_{i,j} \binom{a_{n-3}}{i} \begin{bmatrix} a_{n-2} - ci \\ a_{n-2} - ci - e_2 \end{bmatrix} \begin{bmatrix} -a_{n-3} + e_2 c \\ -a_{n-3} + e_2 c - e_1 + j \end{bmatrix} y^{a_{n-2} - c(e_1 + i - j) - e_2}. \end{aligned}$$

On the other hand, Lemma 14 implies that

$$\begin{aligned}
 (2.9) \quad & \frac{1}{e_2!} \frac{d^{e_2}}{dy^{e_2}} [(1 + y^{-c})^{a_{n-3}} y^{a_{n-2}}] \\
 &= (1 + y^{-c})^{a_{n-3}} \binom{a_{n-2}}{e_2} y^{a_{n-2}-e_2} \\
 &+ \sum_{k=1}^{e_2} \sum_{i=1}^k \sum_{\substack{j_1 + \dots + j_i = k \\ j_1, \dots, j_i \geq 1}} \binom{a_{n-3}}{i} (1 + y^{-c})^{a_{n-3}-i} \begin{bmatrix} -c \\ -c-j_1 \end{bmatrix} \cdots \begin{bmatrix} -c \\ -c-j_i \end{bmatrix} y^{-ci-k} \binom{a_{n-2}}{e_2-k} y^{a_{n-2}-e_2+k}.
 \end{aligned}$$

Combining (2.8) and (2.9), we get

$$\begin{aligned}
 & \sum_{i,j} \binom{a_{n-3}}{i} \begin{bmatrix} a_{n-2} - ci \\ a_{n-2} - ci - e_2 \end{bmatrix} \begin{bmatrix} -a_{n-3} + e_2 c \\ -a_{n-3} + e_2 c - e_1 + j \end{bmatrix} y^{a_{n-2}-c(e_1+i-j)-e_2} \\
 &= (1 + y^{-c})^{ce_2} \binom{a_{n-2}}{e_2} y^{a_{n-2}-e_2} \\
 &+ \sum_{k=1}^{e_2} \sum_{i=1}^k \sum_{\substack{j_1 + \dots + j_i = k \\ j_1, \dots, j_i \geq 1}} \binom{a_{n-3}}{i} (1 + y^{-c})^{ce_2-i} \begin{bmatrix} -c \\ -c-j_1 \end{bmatrix} \cdots \begin{bmatrix} -c \\ -c-j_i \end{bmatrix} \binom{a_{n-2}}{e_2-k} y^{a_{n-2}-e_2-ci}.
 \end{aligned}$$

Comparing the coefficients of $y^{a_{n-2}-ce_1-e_2}$ in both sides, we obtain

$$\begin{aligned}
 (2.10) \quad & \sum_i \binom{a_{n-3}}{i} \begin{bmatrix} a_{n-2} - ci \\ a_{n-2} - ci - e_2 \end{bmatrix} \begin{bmatrix} -a_{n-3} + e_2 c \\ -a_{n-3} + e_2 c - e_1 + i \end{bmatrix} \\
 &= \binom{ce_2}{e_1} \binom{a_{n-2}}{e_2} + \sum_{k=1}^{e_2} \sum_{i=1}^k \sum_{\substack{j_1 + \dots + j_i = k \\ j_1, \dots, j_i \geq 1}} \binom{a_{n-3}}{i} \binom{ce_2-i}{e_1-i} \begin{bmatrix} -c \\ -c-j_1 \end{bmatrix} \cdots \begin{bmatrix} -c \\ -c-j_i \end{bmatrix} \binom{a_{n-2}}{e_2-k}.
 \end{aligned}$$

Then the desired statement follows from Lemma 13. \square

Now the EC-polynomial is expected to come into play.

Conjecture 16. *Assume that $b = c \geq 2$. If $e_1 < c$ and $n \geq 4$, then the Euler-Poincaré characteristic of $\text{Gr}_{(e_1, e_2)}(M(n))$ is equal to*

$$(2.11) \quad \frac{1}{(e_2!)^2} \sum_{z_1 + \dots + z_{e_2} = e_1} \binom{c}{z_1} \cdots \binom{c}{z_{e_2}} EC_{e_2}(\{z_1, \dots, z_{e_2}\}; -a_{n-2}, -a_{n-3}, c).$$

To prove Conjecture 16, one needs to show that (2.6) = (2.11). In fact, the case $e_2 \leq 2$ is elementary, from which the author guessed the general case and checked (2.6) = (2.11) when $e_2 \leq 5$.

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